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Vorwort

Das Tätigkeitsfeld des Fraunhofer-Instituts für Techno- und Wirtschaftsmathematik ITWM umfasst anwendungsnahe Grundlagenforschung, angewandte Forschung sowie Beratung und kundenspezifische Lösungen auf allen Gebieten, die für Techno- und Wirtschaftsmathematik bedeutsam sind.

In der Reihe »Berichte des Fraunhofer ITWM« soll die Arbeit des Instituts kontinuierlich einer interessierten Öffentlichkeit in Industrie, Wirtschaft und Wissenschaft vorgestellt werden. Durch die enge Verzahnung mit dem Fachbereich Mathematik der Universität Kaiserslautern sowie durch zahlreiche Kooperationen mit internationalen Institutionen und Hochschulen in den Bereichen Ausbildung und Forschung ist ein großes Potenzial für Forschungsberichte vorhanden. In die Berichtreihe werden sowohl hervorragende Diplom- und Projektarbeiten und Dissertationen als auch Forschungsberichte der Institutsmitarbeiter und Institutsgäste zu aktuellen Fragen der Techno- und Wirtschaftsmathematik aufgenommen.

Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

On convergence of a discrete problem describing transport processes in the pressing section of a paper machine including dynamic capillary effects: one-dimensional case

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Abstract

This work presents a proof of convergence of a discrete solution to a continuous one. At first, the continuous problem is stated as a system of equations which describe filtration process in the pressing section of a paper machine. Two flow regimes appear in the modeling of this problem. The model for the saturated flow is presented by the Darcy's law and the mass conservation. The second regime is described by the Richards approach together with a dynamic capillary pressure model. The finite volume method is used to approximate the system of PDEs. Then the existence of a discrete solution to proposed finite difference scheme is proven. Compactness of the set of all discrete solutions for different mesh sizes is proven. The main Theorem shows that the discrete solution converges to the solution of continuous problem. At the end we present numerical studies for the rate of convergence.

Keywords: saturated and unsaturated fluid flow in porous media, Richards' approach, dynamic capillary pressure, finite volume methods, convergence of approximate solution

1 Introduction

Paper production is a challenging problem, which attracts attention of many scientists. They try to simulate and investigate the paper making process. Presented here studies are performed for mathematical models describing water filtration in the pressing section of a paper machine. This part of the production process provides the dewatering of the paper layer by the use of fabric, i.e. press felts, which absorb the water during pressing (see [10]).

One of the main challenges of the pressing process modeling is the study of regimes leading to appearance of the fully saturated zones, as a result we have to deal with a free boundary problem. There exist theoretical studies which investigate the convergence of discrete solution for free boundary problems describing various applications such as fluid flow in porous media, obstacle problems and elastic problems (see [2, 4] and references therein). In this work we are concerned with a proof of convergence for the system of equations describing water flow in the pressing section, with minimal restrictions on the input data. Our theoretical studies are, in particular, motivated by a need for a better understanding of, and a more stable background for the results from our computational experiments, which we perform for industry relevant regimes of the pressing process.

The mathematical model for the flow in the pressing section of a paper machine which is presented here includes into consideration the dynamic capillary pressure. To model this effect we choose the dynamic capillary pressure–saturation relation proposed by Hassanizadeh and co-workers [5, 6, 7]. In domain with unsaturated water flow we obtain a system of two nonlinear equations, which makes the theoretical studies more complex than in case of standard (steady) capillary pressure–saturation relation. There are some theoretical studies for the flow model with the dynamic capillary effect. They deal with existence and uniqueness of the solution (see [9] and references therein). As opposed to our work, they have considered a time-dependent problem with the dynamic capillary pressure–saturation relation including partial derivative w.r.t. time. In our case, due to specificity of the pressing process we are concerned with a steady-state problem with the dynamic capillary pressure–saturation relation depending on partial derivative w.r.t. space coordinate. We are not aware of theoretical studies which deal with this kind of problems.

In short, the objectives of this paper are to present an one-dimensional continuous model, which includes all features discussed above, together with numerical algorithm and to study theoretically convergence of the solution of the discrete problem to the solution of the continuous problem. The one-dimensional continuous model, which describes water flow in the pressing section is presented in Section 2. In Section 3, the nonlinear finite difference scheme and its implementation algorithm are introduced. The main part of this paper is the theoretical existence and convergence studies which are presented in Section 4. Some numerical tests are developed in Section 5. Final conclusions are presented in Section 6.

2 Mathematical Model

When one models the pressing section of a paper machine it is important to evaluate fully saturated zones. Therefore, one has to account for two possible flow regimes inside the computational domain. Let us assume that computational domain Ω (see Fig. 1) is divided into two subdomains: in Ω_1, single-phase (water) flow takes place, and in Ω_2, two-phase flow occurs.

Since the aim of this work is to investigate one-dimensional model, we consider a case when the computational domain Ω is composed of one layer. Let us also assume that this layer is transported through the press nips from the left to the right with velocity $\mathbf{V}_{s,in}$ measured in $[m/s]$ as indicated in Fig. 1.

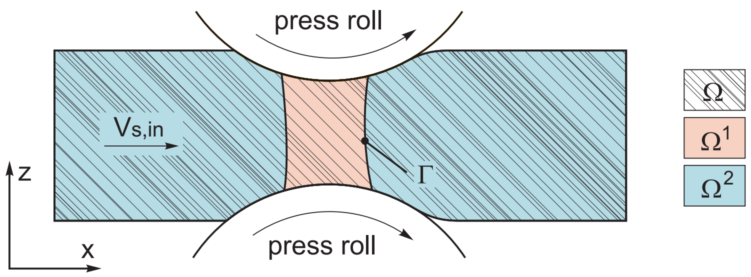

Fig. 1: Computational domain Ω with marked subdomains Ω_1 and Ω_2

A paper machine works in a non-stop regime in several days. Therefore, we are interested in a steady-state solution and all derivatives w.r.t. time are considered to be zero.

The first regime is a single-phase flow model. We describe it with the help of mass conservation equation for the water phase and Darcy's law in the case of moving porous media and neglected gravity term (for more details see [1]):

$$\operatorname{div}(\phi \rho_w \mathbf{V}_w) = 0, \quad x \in \Omega_1,$$

$$\phi(\mathbf{V}_w - \mathbf{V}_s) = -\frac{\mathbf{K}}{\mu_w} \operatorname{grad} p_w, \quad x \in \Omega_1,$$

where ϕ ($[-]$) is the porosity, ρ_w is the density of water measured in $[kg/m^3]$, $\mathbf{V}_w$ is the velocity of water in $[m/s]$, $\mathbf{V}_s$ is the velocity of solid in $[m/s]$, μ_w is the viscosity of the water in $[Pa \cdot s]$, $\mathbf{K}$ is the intrinsic permeability tensor in $[m^2]$, which we assume to be diagonal, p_w is the pressure of water in $[Pa]$. In the following all vectors and tensors will be written in bold fonts.

The second regime is a two-phase flow, which is simulated using Richards' assumptions, the mass conservation equation for water phase, the Darcy law and the dynamic capillary pressure-saturation relation derived by Hassanizadeh and Gray [5, 6, 7] (for more detailed explanations see [8]):

$$\operatorname{div}(\phi S \rho_w \mathbf{V}_w) = 0, \quad x \in \Omega_2,$$

$$\phi S(\mathbf{V}_w - \mathbf{V}_s) = -\frac{k_{rw}}{\mu_w} \mathbf{K} \operatorname{grad} p_w, \quad x \in \Omega_2,$$

$$p_w + p_c^{stat}(S) = \tau \mathbf{V}_s \,\mathrm{grad}\, S, \quad x \in \Omega_2,$$

where S ($[-]$) is the saturation of water, k_{rw} ($[-]$) is the relative permeability, p_c^{stat} is the static capillary pressure-saturation relation, τ is the material coefficient measured in $[Pa \cdot s]$.

In general τ may depend on saturation and other parameters, but in these studies we are concerned only with case when τ is a constant. We also remark that case when $\tau = 0$ leads to the standard model with static capillary pressure. This model was investigated theoretically in [3].

Remark 1. *To simplify the following presentation we assume that p_c^{stat} depends only on S. But in general it should be assumed that $p_c^{stat} = p_c^{stat}(S, \phi(x))$. We note that main steps of the proof remain valid also for this more general case.*

We introduce operator $[f]_\Gamma$ which indicates a jump of a function f across some interface Γ:

$$[f]_\Gamma = \lim_{t \to \Gamma + 0} f(t) - \lim_{t \to \Gamma - 0} f(t).$$

Then, the interfacial conditions, the continuity of water pressures and the continuity of normal fluxes, between domains Ω_1 and Ω_2 yield:

$$[p]_\Gamma = 0, \quad [\mathbf{J} \cdot \mathbf{n}]_\Gamma = 0, \tag{1}$$

where Γ is the common boundary between the domains Ω_1 and Ω_2, $\mathbf{n}$ is the unit normal vector to Γ, $\mathbf{J}$ is the water flux, which is defined as:

$$\mathbf{J} = \begin{cases} -\frac{\mathbf{K}}{\mu_w} \,\mathrm{grad}\, p_w + \phi \mathbf{V}_s & \text{for } x \in \Omega_1; \\ -\frac{\mathbf{K} k_{rw}}{\mu_w} \,\mathrm{grad}\, p_w + \phi S \mathbf{V}_s & \text{for } x \in \Omega_2. \end{cases}$$

To simplify the notations from now on instead of p_w, μ_w and k_{rw} we are going to use more simple notations p, μ and k.

We obtain the one-dimensional model by averaging the two-dimensional model in vertical direction. Therefore, a thickness of the layer $d(x)$ is included into the final model (see [8, 11] for details). Assuming that the water is incompressible, we have:

$$-\frac{\partial}{\partial x}\left(d(x) \frac{K(\phi(x))}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial x}\left(d(x) \phi(x) V_s \right) = 0, \quad x \in \Omega_1, \tag{2}$$

$$-\frac{\partial}{\partial x}\left(d(x) \frac{k(S)}{\mu} K(\phi(x)) \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial x}\left(d(x) \phi(x) V_s S \right) = 0, \quad x \in \Omega_2, \tag{3}$$

$$p + p_c^{stat}(S) = \tau V_s \frac{\partial S}{\partial x} \quad x \in \Omega_2, \tag{4}$$

where $d(x)$ is the thickness of the layer measured in $[m]$, V_s is considered as x-component of vector $\mathbf{V}_s$, K is considered as xx-component of tensor $\mathbf{K}$. The computational domain Ω is assumed to be an open interval $(0, 1)$.

Let us define functions $b(x) = d(x) K(\phi(x))/\mu$, $q(x) = d(x) \phi(x) V_s$ and $c = \tau V_s = const$. Then, the nonlinear system of Eqs. (2–4) can be rewritten as:

$$-\frac{\partial}{\partial x}\left(b(x) \frac{\partial p}{\partial x} \right) + \frac{\partial q(x)}{\partial x} = 0, \quad x \in \Omega_1 \tag{5}$$

$$-\frac{\partial}{\partial x}\left(b(x)k(S)\frac{\partial p}{\partial x}\right)+\frac{\partial(q(x)S)}{\partial x}=0,\ \ x\in\Omega_2 \tag{6}$$

$$p+p_c^{stat}(S)=c\frac{\partial S}{\partial x},\ \ x\in\Omega_2. \tag{7}$$

The boundary conditions yield:

$$p(0)=-p_c^{stat}(C_0), \tag{8}$$

$$\left.\frac{\partial p}{\partial x}\right|_{x=1}=0, \tag{9}$$

$$S(0)=C_0. \tag{10}$$

The interfacial conditions (1) have also to be satisfied for the one-dimensional water flux defined by:

$$J=\begin{cases}-b(x)\frac{\partial p}{\partial x}+\phi V_s & \text{for } x\in\Omega_1;\\ -b(x)k\frac{\partial p}{\partial x}+\phi SV_s & \text{for } x\in\Omega_2.\end{cases} \tag{11}$$

Let us impose the following assumptions on the input data:

Assumption 1.

(a) $b(x)\in C([0,1])$, $b(x)>0$;

(b) $q(x)\in C([0,1])$, $q(x)\geq 0$;

(c) $k(S)\in C([S_*,1])$, $k:[S_*,1]\to[k_*,1]$ *is an increasing function, where* $k_*\in\mathbb{R}$ *and* $k_*>0$;

(d) $C_0\in(S_*,1)$;

(e) $c\in\mathbb{R}$, $c>0$;

(f) $p_c^{stat}\in C^1([S_*,1])$, $p_c^{stat}:[S_*,1]\leftrightarrow[p_*,p^*]$ *is a decreasing function, where* $S_*\in\mathbb{R}$ *and* $S_*>0$.

These assumptions are made taking into account input data which is typically used in the real numerical experiments.

Taking into account imposed assumptions we can reformulate problem (5–7) in the following way:

$$-\frac{\partial}{\partial x}\left(b(x)k(S)\frac{\partial p}{\partial x}\right)+\frac{\partial(q(x)S)}{\partial x}=0,\ \ x\in\Omega, \tag{12}$$

$$\hat{c}(S)(p+p_c^{stat}(S))=\frac{\partial S}{\partial x},\ \ x\in\Omega, \tag{13}$$

where p is the water pressure in computational domain Ω and function $\hat{c}(S)$ takes the form:

$$\hat{c}(S)=\begin{cases}1/c & \text{for } S\in(S_*,1);\\ 0 & \text{for } S\notin(S_*,1).\end{cases} \tag{14}$$

Due to Assumption 1, we notice that Eq. (12) coincides with Eq. (5) in the domain Ω_1 and with Eq. (6) in the domain Ω_2. Continuity of the pressure p

in whole domain Ω follows from the definition of the non-linear convection–diffusion Eq. (12). Continuity of the normal fluxes directly follows from integration of Eq. (12) in a small interval which contains the boundary between Ω_1 and Ω_2.

Eq. (13) with (14) transforms automatically into Eq. (7) in the domain Ω_2. Let us prove that in the domain Ω_1 one of the following equations are satisfied:

$$S = S_*, \quad S = 1.$$

For this we are going to show that solution to (13,14) is bounded and belongs to interval $[S_*, 1]$. Integrating (13) over interval $(0, x)$ for some $x \in (0, 1)$ and then finding $|S(x) - S(y)|$ we can show that solution to (13,14), S, is a continuous function.

Let us assume that there exists $\tilde{x} \in \Omega$ such that $S(\tilde{x}) > 1$. Since the function S is continuous there exists $y \in (0, \tilde{x})$ such that $S(y) = 1$ and $S(x) > 1$ for all $x \in (y, \tilde{x}]$. Then, we have:

$$\begin{aligned} S(\tilde{x}) &= C_0 + \int_0^y \hat{c}(S)(p + p_c^{stat}(S))dt + \int_y^{\tilde{x}} \hat{c}(S)(p + p_c^{stat}(S))dt \\ &= S(y) + \int_y^{\tilde{x}} \hat{c}(S)(p + p_c^{stat}(S))dt = S(y). \end{aligned}$$

Hence, we have obtained a contradiction $S(\tilde{x}) = S(y)$, which proves that $S \leq 1$. Using the same approach it can be proven that $S \geq S_*$. Thus, system of Eqs. (13,14) guarantees that solution S is in $[S_*, 1]$.

Remark 2. *The new model (12–14) also contains a fictitious regime when saturation is equal to S_*. This regime is required for the completeness of the model and the following proof. From the physical point of view instead of Eqs. (12–14) in this case we formulate the following equations:*

$$p = -p^*, \quad S = S_*.$$

On practice, in all one-dimensional numerical experiments we assume that we do not have regions where single-phase air flow occurs.

In order to simplify notations, we apply variable transformation $p = y - p_c^{stat}(C_0)$, then we obtain the following nonlinear boundary value problem:

$$-\frac{\partial}{\partial x}\left(b(x)k(S)\frac{\partial y}{\partial x}\right) + \frac{\partial (q(x)S)}{\partial x} = 0, \quad x \in (0, 1), \tag{15}$$

$$\hat{c}(S)(y + g(S)) = \frac{\partial S}{\partial x}, \quad x \in (0, 1], \tag{16}$$

$$y(0) = 0, \tag{17}$$

$$\left.\frac{\partial y}{\partial x}\right|_{x=1} = 0, \tag{18}$$

$$S(0) = C_0, \tag{19}$$

where $g(S) = p_c^{stat}(S) - p_c^{stat}(C_0)$.

Let $H^1_{0-}((0, 1))$ be the subspace of $H^1((0, 1))$ satisfying

$$H^1_{0-}((0, 1)) := \{\, f \in H^1((0, 1)) \,|\, f(0) = 0\}.$$

Then, we consider the weak formulation of problem (15–19):
find $y \in H^1_{0-}((0,1))$ and $S \in L_2((0,1))$ such that

$$\int_0^1 b(x)k(S)\frac{\partial y}{\partial x}\frac{\partial \varphi}{\partial x}dx - \int_0^1 q(x)S\frac{\partial \varphi}{\partial x}dx + q(1)S(1)\varphi(1) = 0, \tag{20}$$

$$-\int_0^1 \hat{c}(S)\,(y + g(S))\,\varphi dx - \int_0^1 S\frac{\partial \varphi}{\partial x}dx + S(1)\varphi(1) = 0, \tag{21}$$

for all $\varphi \in C^\infty([0,1])$ such that $\varphi(0) = 0$.

In order to prove the main convergence theorem we will assume that the following non-degeneracy property is satisfied.

Assumption 2.
For any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that:

$$\text{meas}\left(\{x \in \Omega : S \in (S_*, S_* + \delta_\epsilon) \cup (1 - \delta_\epsilon, 1)\}\right) \leq \epsilon. \tag{22}$$

3 Discretization

Let us introduce a mesh on $(0,1)$.

Definition 1. *The mesh on $(0,1)$ denoted by $\mathcal{T}$ is given by a family $(\mathcal{K}_i)_{i=\overline{0,N}}$, $N \in \mathbb{N}^+$ such that:*

$$\begin{aligned}
\mathcal{K}_0 &= (x_0, x_{\frac{1}{2}}], \\
\mathcal{K}_i &= (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \quad i = \overline{1, N-1}, \\
\mathcal{K}_N &= (x_{N-\frac{1}{2}}, x_N)
\end{aligned}$$

and families:

$$\begin{aligned}
\bar{\omega}_1 &= \{x_i = ih,\ i = \overline{0,N}\}, \\
\bar{\omega}_2 &= \{x_{i+\frac{1}{2}} = \left(i + \frac{1}{2}\right)h,\ i = \overline{0, N-1}\},
\end{aligned}$$

where $h = 1/N$ is the size of the mesh.

In order to discretize Eq. (15) and boundary conditions (17), (18) let us consider the following finite volume scheme:

$$y_0 = 0, \tag{23}$$

$$\begin{aligned}
-b_{i+\frac{1}{2}}k_{i+\frac{1}{2}}\frac{y_{i+1} - y_i}{h} &+ b_{i-\frac{1}{2}}k_{i-\frac{1}{2}}\frac{y_i - y_{i-1}}{h} \\
&+ (q_{i+\frac{1}{2}}S_{i+\frac{1}{2}} - q_{i-\frac{1}{2}}S_{i-\frac{1}{2}}) = 0, \quad i = \overline{1, N-1},
\end{aligned} \tag{24}$$

$$b_{N-\frac{1}{2}}k_{N-\frac{1}{2}}\frac{y_N - y_{N-1}}{h} + (q_N S_N - q_{N-\frac{1}{2}}S_{N-\frac{1}{2}}) = 0, \tag{25}$$

where

$$k_{i+\frac{1}{2}} = k(S_{i+\frac{1}{2}}), \qquad b_{i+\frac{1}{2}} = b(x_{i+\frac{1}{2}}), \qquad q_{i+\frac{1}{2}} = q(x_{i+\frac{1}{2}}). \tag{26}$$

Finite volume scheme for Eq. (16) with boundary condition (19) is introduced using an intermediate value of saturation. For each spatial step i we split the discrete algorithm into two steps. In the first step, a prediction of water saturation value $\hat{S}_{i+\frac{1}{2}}$ for any $i = \overline{0, N-1}$ or $\hat{S}_N$ is computed by solving the appropriate nonlinear equation:

$$S_0 = C_0, \tag{27}$$

$$\frac{1}{c}\left(y_0 + \tilde{g}(\hat{S}_{\frac{1}{2}})\right) = \frac{2}{h}(\hat{S}_{\frac{1}{2}} - S_0), \tag{28}$$

$$\frac{1}{c}\left(y_i + \tilde{g}(\hat{S}_{i+\frac{1}{2}})\right) = \frac{1}{h}(\hat{S}_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}), \quad i = \overline{1, N-1} \tag{29}$$

$$\frac{1}{c}\left(y_N + \tilde{g}(\hat{S}_N)\right) = \frac{2}{h}(\hat{S}_N - S_{N-\frac{1}{2}}), \tag{30}$$

where

$$\tilde{g}(S) = \begin{cases} g(S_*) & \text{for } S < S_*, \\ g(S) & \text{for } S_* \leq S \leq 1, \\ g(1) & \text{for } S > 1. \end{cases} \tag{31}$$

At the second step, this value is corrected with the help of a simple restriction operator:

$$S_i = \begin{cases} S_* + \eta & \text{for } \hat{S}_i < S_* + \eta; \\ \hat{S}_i & \text{for } S_* + \eta \leq \hat{S}_i \leq 1 - \eta; \\ 1 - \eta & \text{for } \hat{S}_i > 1 - \eta; \end{cases} \tag{32}$$

for all $i = \{\overline{\frac{1}{2}, N - \frac{1}{2}}, N\}$. Here $\eta > 0$ is some small value which satisfies $\eta \to 0$ as $h \to 0$.

We note here, that the correction step in an implicit way defines the discrete analog of the function $\hat{c}(S)$.

4 Proof of Convergence

To prove the convergence of discrete solution (23–32) to continuous solution of (20,21), first we consider Eqs. (23–26) separately from Eqs. (27–32). In the following two lemmas we prove existence of solutions to each of these problems.

Let us introduce the following notation:

$$S_{-\frac{1}{2}} = S_0, \quad S_{N+\frac{1}{2}} = S_N.$$

Lemma 1. *Let Assumption 1 be satisfied and let $\mathcal{T}$ be the mesh on $(0,1)$ (see Definition 1). Let $\mathbf{S} = (S_{-\frac{1}{2}}, S_{\frac{1}{2}}, \ldots, S_{N+\frac{1}{2}})^T \in \mathbb{R}^{N+2}$ be some given vector, such that $S_{i-\frac{1}{2}} \in [S_* + \eta, 1 - \eta]$ for all $i = \overline{0, N+1}$. Then, there exists a unique solution to (23–26), $\mathbf{y} = (y_0, y_1, \ldots, y_N)^T \in \mathbb{R}^{N+1}$, such that:*

$$\sum_{i=0}^{N-1} \frac{(y_{i+1} - y_i)^2}{h} \leq C_1^2 = \left(\frac{2q^*}{b_* k_*}\right)^2,$$

where $q_* \leq q_{i+\frac{1}{2}} \leq q^*$ and $b_* \leq b_{i+\frac{1}{2}} \leq b^*$ for all $i = \overline{0, N-1}$.

Proof. Following the proof of Lemma 1 and Lemma 2 from [3] we obtain the required results. □

For any given vector $\mathbf{v} = (v_{-\frac{1}{2}}, v_{\frac{1}{2}}, ..., v_{N+\frac{1}{2}}) \in \mathbb{R}^{N+2}$ we introduce the following seminorm:

$$\|D\mathbf{v}\|_{L_2((0,1))} = \left(\sum_{i=0}^{N} \frac{(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}})^2}{h_i} \right)^{\frac{1}{2}},$$

where $h_0 = h/2$, $h_i = h$ for all $i = \overline{1, N-1}$, $h_N = h/2$.

Lemma 2. *Let Assumption 1 be satisfied and let $\mathcal{T}$ be the mesh on $(0,1)$ (see Definition 1). Let $\mathbf{y} = (y_0, y_1, \dots, y_N)^T \in \mathbb{R}^{N+1}$ be some given vector, such that $|y_i| \le C_1$ for all $i = \overline{0, N}$. Then, there exist a solution to (27–32), $\mathbf{S} = (S_{-\frac{1}{2}}, S_{\frac{1}{2}}, \dots, S_{N+\frac{1}{2}})^T \in \mathbb{R}^{N+2}$, such that:*

$$S_{i-\frac{1}{2}} \in [S_* + \eta, 1 - \eta], \quad i = \overline{0, N+1} \tag{33}$$

and

$$\|D\mathbf{S}\|_{L_2((0,1))} \le C_2 = \frac{C_1 + g^*}{c}, \tag{34}$$

where $g^* = p^* - p_c^{stat}(C_0)$.

Proof. The system (28–30) can be considered as a Cauchy problem with initial condition (27). Hence we can solve these equations sequentially. At first, let us consider Eq. (28) in the following form:

$$\hat{S}_{\frac{1}{2}} = S_{-\frac{1}{2}} + \frac{h}{2c} \left(y_0 + \tilde{g}(\hat{S}_{\frac{1}{2}}) \right). \tag{35}$$

In Eq. (35), $S_{-\frac{1}{2}}$ is given by (27). We denote the right-hand side of Eq. (35) as $G(\hat{S}_{\frac{1}{2}})$. We notice that G is a continuous function of $\hat{S}_{\frac{1}{2}}$. It is easy to see, that:

$$|G(v)| \le C_0 + \frac{h}{2c}(C_1 + g^*),$$

for any given v. It means $G(B_r) \subset B_r$, where $B_r \subset \mathbb{R}$ is a closed ball with radius $r = C_0 + \frac{h}{2c}(C_1 + g^*)$ and center 0. Using Brouwer's fixed point theorem we conclude that G has at least one fixed point in B_r, which is a solution of (35).

After the correction step (32) for value $S_{\frac{1}{2}}$ we have:

$$S_{\frac{1}{2}} \in [S_* + \eta, 1 - \eta].$$

Using the same technique we can prove that $S_{i-\frac{1}{2}}$ for $i = \overline{2, N+1}$ exists and it is bounded as follows:

$$S_{i-\frac{1}{2}} \in [S_* + \eta, 1 - \eta].$$

It remains to prove estimate (34) for $\|D\mathbf{S}\|_{L_2((0,1))}$. Using Eqs. (28–32) we obtain:

$$\frac{\left| S_{\frac{1}{2}} - S_{-\frac{1}{2}} \right|}{h/2} \le \frac{\left| \hat{S}_{\frac{1}{2}} - S_{-\frac{1}{2}} \right|}{h/2} \le \frac{C_1 + g^*}{c}; \tag{36}$$

$$\frac{\left| S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}} \right|}{h} \le \frac{\left| \hat{S}_{i+\frac{1}{2}} - S_{i-\frac{1}{2}} \right|}{h} \le \frac{C_1 + g^*}{c}, \quad i = \overline{1, N-1}; \tag{37}$$

$$\frac{\left| S_{N+\frac{1}{2}} - S_{N-\frac{1}{2}} \right|}{h/2} \le \frac{\left| \hat{S}_{N+\frac{1}{2}} - S_{N-\frac{1}{2}} \right|}{h/2} \le \frac{C_1 + g^*}{c}. \tag{38}$$

Then, for $\|D\mathbf{S}\|^2_{L_2((0,1))}$ we have:

$$\begin{aligned}\|D\mathbf{S}\|^2_{L_2((0,1))} &\le \left(\frac{C_1+g^*}{c}\right)^2\frac{h}{2}+\sum_{i=1}^{N-1}\left(\frac{C_1+g^*}{c}\right)^2 h+\left(\frac{C_1+g^*}{c}\right)^2\frac{h}{2}\\ &= \left(\frac{C_1+g^*}{c}\right)^2 h\left(\frac{1}{2}+(N-1)+\frac{1}{2}\right)=\left(\frac{C_1+g^*}{c}\right)^2.\end{aligned}$$

□

Lemma 3. (Existence) *Let Assumption 1 be satisfied and let $\mathcal{T}$ be the mesh on $(0,1)$ (see Definition 1). Then, there exist a pair of vectors $\mathbf{y}=(y_0,y_1,\dots,y_N)^T\in\mathbb{R}^{N+1}$ and $\mathbf{S}=(S_{-\frac{1}{2}},S_{\frac{1}{2}},\dots,S_{N+\frac{1}{2}})^T\in\mathbb{R}^{N+2}$, which is solution to the system of Eqs. (23–32).*

Proof. Let us consider auxiliary system of equations obtained from system (23–26) by replacing $\mathbf{S}$ with a vector $\mathbf{v}$ and from system (27–32) by replacing $\mathbf{y}$ with a vector $\mathbf{u}$. The vectors $\mathbf{v}$ and $\mathbf{u}$ satisfy:

$$\mathbf{u}=(u_0,u_1,\dots,u_N)^T\in\mathbb{R}^{N+1},\ \ |u_i|\le C_1 \text{ for all } i=\overline{0,N}; \tag{39}$$

$$\mathbf{v}=(v_{-\frac{1}{2}},v_{\frac{1}{2}},\dots,v_{N+\frac{1}{2}})^T\in\mathbb{R}^{N+2},\ \ v_{i-\frac{1}{2}}\in[S_*+\eta,1-\eta] \text{ for all } i=\overline{0,N+1}. \tag{40}$$

Then it follows from Lemmas 1,2 that there exists an operator $T:\mathbb{R}^{2N+3}\to\mathbb{R}^{2N+3}$ such that:

$$\chi=T(\theta), \tag{41}$$

where $\chi=(\mathbf{y},\mathbf{S})^T\in\mathbb{R}^{2N+3}$ and $\theta=(\mathbf{u},\mathbf{v})^T\in\mathbb{R}^{2N+3}$. Let us assume that operator T is continuous (we will prove this property later). For any given vector $\psi=(\mathbf{x},\mathbf{z})^T\in\mathbb{R}^{2N+3}$ we define the following norm:

$$\|\psi\|_{L_2((0,1))}=\left(\sum_{i=0}^{N}x_i^2h_i+\sum_{i=0}^{N}z^2_{i+\frac{1}{2}}h_i\right)^{\frac{1}{2}}, \tag{42}$$

where $h_0=h_N=h/2$, $h_i=h$ for all $i=\overline{1,N-1}$ and:

$$\mathbf{x}=(x_0,x_1,\dots,x_N)^T\in\mathbb{R}^{N+1},\quad x_0=0; \tag{43}$$

$$\mathbf{z}=(z_{-\frac{1}{2}},z_{\frac{1}{2}},\dots,z_{N+\frac{1}{2}})^T\in\mathbb{R}^{N+2},\quad z_{-\frac{1}{2}}=C_0. \tag{44}$$

Then, for any $\theta=(\mathbf{u},\mathbf{v})^T$ with $\mathbf{u}$ and $\mathbf{v}$, which satisfy (39,40), it follows that

$$\|\theta\|_{L_2((0,1))}\le C_3=\left(C_1^2+1\right)^{\frac{1}{2}}.$$

Due to the properties of the finite volume scheme (23–32), we also have that:

$$\|T(\theta)\|_{L_2((0,1))}=\|\chi\|_{L_2((0,1))}\le C_3.$$

Then using Brouwer's fixed point theorem we conclude that there exists a solution of the system of Eqs. (23–32).

In oder to apply the fixed point theorem we have to show that operator T is continuous. We notice that operator T consists of two operators. The first operator $\mathbf{y} = T_y(\mathbf{S})$ is defined by system of Eqs. (23–26) with some given vector $\mathbf{S}$. Continuity of this operator is a standard result from theory of finite volume schemes and it follows from the coefficient stability of elliptic operators.

The second operator $\mathbf{S} = T_S(\mathbf{y})$ is defined by (27–32) with some given vector $\mathbf{y}$. Let us prove that T_S is continuous, if Assumptions 1 are satisfied. Let us consider two different input vectors $\mathbf{y}$ and $\mathbf{u}$ and denote the corresponding solutions by $\mathbf{S}^y = T_S(\mathbf{y})$ and $\mathbf{S}^u = T_S(\mathbf{u})$. We want to prove that for any $\epsilon > 0$ there exists $\delta = \delta_\epsilon > 0$ such that:

$$\|\mathbf{y} - \mathbf{u}\|_{1,L_2((0,1))} < \delta_\epsilon \implies \|\mathbf{S}^y - \mathbf{S}^u\|_{2,L_2((0,1))} < \epsilon, \tag{45}$$

where the norms are introduced as:

$$\|\mathbf{x}\|_{1,L_2((0,1))} = \left(\sum_{i=0}^{N} x_i^2 h_i\right)^{\frac{1}{2}}, \quad \|\mathbf{z}\|_{2,L_2((0,1))} = \left(\sum_{i=0}^{N} z_{i+\frac{1}{2}}^2 h_i\right)^{\frac{1}{2}},$$

and $\mathbf{x}$, $\mathbf{z}$ satisfy conditions (43,44). Let us write Eqs. (29), $i = \overline{1, N-1}$ in the following form:

$$\hat{S}^y_{i+\frac{1}{2}} - S^y_{i-\frac{1}{2}} = \frac{h}{c}\tilde{g}(\hat{S}^y_{i+\frac{1}{2}}) + \frac{h}{c} y_i, \tag{46}$$

$$\hat{S}^u_{i+\frac{1}{2}} - S^u_{i-\frac{1}{2}} = \frac{h}{c}\tilde{g}(\hat{S}^u_{i+\frac{1}{2}}) + \frac{h}{c} u_i. \tag{47}$$

Introducing vectors $\hat{e}_{i+\frac{1}{2}} = \hat{S}^y_{i+\frac{1}{2}} - \hat{S}^u_{i+\frac{1}{2}}$, $e_{i-\frac{1}{2}} = S^y_{i-\frac{1}{2}} - S^u_{i-\frac{1}{2}}$, and subtracting Eq. (47) from (46) we get:

$$\hat{e}_{i+\frac{1}{2}} - e_{i-\frac{1}{2}} = \frac{h}{c}\left(\tilde{g}(\hat{S}^y_{i+\frac{1}{2}}) - \tilde{g}(\hat{S}^u_{i+\frac{1}{2}})\right) + \frac{h}{c}(y_i - u_i). \tag{48}$$

Taking into account the definition of function $\tilde{g}$ in (31), we get the estimate:

$$\tilde{g}(\hat{S}^y_{i+\frac{1}{2}}) - \tilde{g}(\hat{S}^u_{i+\frac{1}{2}}) = g'(S^\xi_{i+\frac{1}{2}})\theta \hat{e}_{i+\frac{1}{2}},$$

where $S_* \le S^\xi_{i+\frac{1}{2}} \le 1$ and $0 \le \theta \le 1$. Since the function g is a decreasing function, then $g'(S^\xi_{i+\frac{1}{2}}) < 0$.

It follows from the definition of the restriction operator (32) that $|e_{i+\frac{1}{2}}| \le |\hat{e}_{i+\frac{1}{2}}|$. Then, using (48), we get:

$$\left|e_{i+\frac{1}{2}}\right| \le \left|\hat{e}_{i+\frac{1}{2}}\right| \le \left|1 - \frac{h}{c} g'(S^\xi_{i+\frac{1}{2}})\theta\right| \left|\hat{e}_{i+\frac{1}{2}}\right| \le \left|e_{i-\frac{1}{2}}\right| + \frac{h}{c}|y_i - u_i|. \tag{49}$$

Similarly, from Eqs. (28,30) we obtain:

$$\left|e_{\frac{1}{2}}\right| \le \frac{h}{2c}\left|y_0 - u_0\right|, \quad \left|e_{N+\frac{1}{2}}\right| \le \left|e_{N-\frac{1}{2}}\right| + \frac{h}{2c}\left|y_N - u_N\right|. \tag{50}$$

Using first inequalities (49,50) and then the Cauchy–Schwartz inequality we have:

$$\left|e_{j+\frac{1}{2}}\right| \le \sum_{i=0}^{N} \frac{h_i}{c}\left|y_i - u_i\right| \le \frac{1}{c}\left(\sum_{i=0}^{N}(y_i - u_i)^2 h_i \sum_{i=0}^{N} h_i\right)^{\frac{1}{2}}$$
$$= \frac{1}{c}\|\mathbf{y} - \mathbf{u}\|_{1,L_2((0,1))}, \quad j = 0, \ldots, N. \tag{51}$$

From (51) we obtain:

$$\|\mathbf{e}\|^2_{2,L_2((0,1))} = \sum_{i=0}^{N} |e_{i+\frac{1}{2}}|^2 h_i \le \frac{1}{c^2}\|\mathbf{y}-\mathbf{u}\|^2_{1,L_2((0,1))},$$

where h_i is defined in (42). Hence statement (45) is proven, therefore operator T is continuous. $\square$

Lemma 4. (Compactness) *Let Assumptions 1 be satisfied and let $\mathcal{T}$ be a mesh on $(0,1)$ (see Definition 1). Let the pair of vectors $\mathbf{y} = (y_0, y_1, \ldots, y_N)^T \in \mathbb{R}^{N+1}$ and $\mathbf{S} = (S_{-\frac{1}{2}}, S_{\frac{1}{2}}, \ldots, S_{N+\frac{1}{2}})^T \in \mathbb{R}^{N+2}$ be a solution to (23–32). Let $y_{\mathcal{T}} : (0,1) \to \mathbb{R}$ be $y_{\mathcal{T}}(x) = y_i$ and let $S_{\mathcal{T}} : (0,1) \to [S_* + \eta, 1-\eta]$ be $S_{\mathcal{T}}(x) = S_{i+\frac{1}{2}}$ for $x \in \mathcal{K}_i$, $i = \overline{0,N}$. Then, the sets $y_{\mathcal{T}}$ and $S_{\mathcal{T}}$ are relatively compact in $L^2((0,1))$. Furthermore, if $y_{\mathcal{T}_n} \to y$ and $S_{\mathcal{T}_n} \to S$ in $L^2((0,1))$ and $h_n \to 0$ as $n \to \infty$, then, $y \in H^1_{0-}((0,1))$ and $S \in H^1((0,1))$.*

Proof. All statements for $y_{\mathcal{T}}$ were proven in Lemma 3 in [3]. Therefore, here we are concerned only with the function $S_{\mathcal{T}}$.

Using Kolmogorov compactness theorem, it is sufficient to show that $S_{\mathcal{T}}$ is relatively compact in $L^2((0,1))$:

- the set $S_{\mathcal{T}}$ is bounded in $L^2(\mathbb{R})$ for all $\mathcal{T}$,
- $\|S_{\mathcal{T}}(\cdot+\nu) - S_{\mathcal{T}}\|_{L^2(\mathbb{R})} \to 0$ as $\nu \to 0$ uniformly.

Step 1. Function $S_{\mathcal{T}}$ can be redefined as $S_{\mathcal{T}}(x) = S_{i+\frac{1}{2}}$ if $x \in \mathcal{K}_i$, $i = \overline{0,N}$, $S_{\mathcal{T}}(x_0) = S_{-\frac{1}{2}}$, otherwise $S_{\mathcal{T}} = 0$. Then, using (33) it follows immediately that the set $S_{\mathcal{T}}$ for all $\mathcal{T}$ is bounded in $L^2(\mathbb{R})$.

Step 2. Let $0 < \nu < 1$. We define $\chi_i : \mathbb{R} \to \mathbb{R}$ for $i = \overline{-1, N+1}$ such that:

$$\begin{aligned}
&\chi_{-\frac{1}{2}}(x) = 1 \text{ if } x_0 \in [x, x+\nu], && \chi_{-\frac{1}{2}}(x) = 0, \text{ otherwise;}\\
&\chi_{i+\frac{1}{2}}(x) = 1, \text{ if } x_{i+\frac{1}{2}} \in [x, x+\nu], && \chi_{i+\frac{1}{2}}(x) = 0, \text{ otherwise, } i = \overline{0, N-1};\\
&\chi_{N+\frac{1}{2}}(x) = 1; \text{ if } x_N \in [x, x+\nu], && \chi_{N+\frac{1}{2}}(x) = 0, \text{ otherwise.}
\end{aligned}$$

Then for all $x \in \mathbb{R}$ we have:

$$\begin{aligned}
&(S_{\mathcal{T}}(x+\nu) - S_{\mathcal{T}}(x))^2 \le \left(S_{-\frac{1}{2}}\chi_{-\frac{1}{2}} + \sum_{i=0}^{N}\left|S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right|\chi_{i-\frac{1}{2}} + S_{N+\frac{1}{2}}\chi_{N+\frac{1}{2}}\right)^2\\
&\le 3S^2_{-\frac{1}{2}}\chi_{-\frac{1}{2}} + 3\left(\sum_{i=0}^{N}\left|S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right|\chi_{i-\frac{1}{2}}\right)^2 + 3S^2_{N+\frac{1}{2}}\chi_{N+\frac{1}{2}}\\
&\le 3S^2_{-\frac{1}{2}}\chi_{-\frac{1}{2}} + 3\left(\sum_{i=0}^{N}\frac{\left(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right)^2}{h_i}\chi_{i-\frac{1}{2}}\right)\left(\sum_{i=0}^{N}h_i\chi_{i-\frac{1}{2}}\right) + 3S^2_{N+\frac{1}{2}}\chi_{N+\frac{1}{2}},
\end{aligned} \tag{52}$$

where $h_0 = h_N = h/2$ and $h_i = h$ for all $i = \overline{1, N-1}$. Integrating (52) over $\mathbb{R}$

we obtain:

$$
\begin{aligned}
\|S_{\mathcal{T}}(\cdot+\nu)-S_{\mathcal{T}}\|^2_{L^2(\mathbb{R})} &\le \int_{\mathbb{R}} 3S^2_{-\frac{1}{2}}\chi_{-\frac{1}{2}}dx+\int_{\mathbb{R}} 3S^2_{N+\frac{1}{2}}\chi_{N+\frac{1}{2}}dx \\
&+\int_{\mathbb{R}} 3\left(\sum_{i=0}^{N}\frac{\left(S_{i+\frac{1}{2}}-S_{i-\frac{1}{2}}\right)^2}{h_i}\chi_{i-\frac{1}{2}}\right)\left(\sum_{i=0}^{N}h_i\chi_{i-\frac{1}{2}}\right)dx \\
&\le 6\nu+3(\nu+2h)\int_{\mathbb{R}}\left(\sum_{i=0}^{N}\frac{\left(S_{i+\frac{1}{2}}-S_{i-\frac{1}{2}}\right)^2}{h_i}\chi_{i-\frac{1}{2}}\right)dx \\
&\le 6\nu+3(\nu+2h)\sum_{i=0}^{N}\frac{\left(S_{i+\frac{1}{2}}-S_{i-\frac{1}{2}}\right)^2}{h_i}\int_{\mathbb{R}}\chi_{i-\frac{1}{2}}dx \\
&\le 6\nu+3\nu(\nu+2h)C_2^2.
\end{aligned}
$$

Since $\nu<1$ and $h<1$ we conclude that:

$$
\|S_{\mathcal{T}}(\cdot+\nu)-S_{\mathcal{T}}\|^2_{L^2(\mathbb{R})}\le \hat{C}\nu,
$$

where $\hat{C}=const>0$. Hence, the second condition of Kolmogorov compactness theorem is proven.

Step 3. Here we want to prove that function $S(x)$ belongs to $H^1((0,1))$. At first let us prove that $(S_{\mathcal{T}_n}(x+h_n)-S_{\mathcal{T}_n}(x))/h_n$ converges to $\partial S/\partial x$ for all $x\in(-\infty,1)$ in a weak sense when $h_n\to 0$ as $n\to\infty$. Let $\varphi\in C_0^\infty((-\infty,1))$ and $\operatorname{supp}\varphi\subset(0,1)$. The discrete function $\varphi_{\mathcal{T}_n}$ is defined in the following way:

$$
\varphi_{\mathcal{T}_n}(x)=\begin{cases}\varphi_i=\varphi(x_i) & \text{if } x\in\mathcal{K}_i,\, i=\overline{0,N};\\ 0, & \text{otherwise.}\end{cases}
$$

Let us redefine function $S_{\mathcal{T}_n}$ such that $S_{\mathcal{T}_n}=S_{N+\frac{1}{2}}$ if $x\in[x_N,x_{N+\frac{1}{2}}]$ and $S_{\mathcal{T}_n}=S_{N+\frac{3}{2}}=S_{N+\frac{1}{2}}$ if $x\in[x_{N+\frac{1}{2}},x_{N+\frac{3}{2}}]$. Then, we have:

$$
\begin{aligned}
\left(\frac{S_{\mathcal{T}_n}(\cdot+h_n)-S_{\mathcal{T}_n}}{h_n},\varphi_{\mathcal{T}_n}\right)_{L_2((-\infty,1))} &= \int_{-\infty}^{1}\frac{S_{\mathcal{T}_n}(x+h_n)-S_{\mathcal{T}_n}(x)}{h_n}\varphi_{\mathcal{T}_n}(x)dx \\
&= \sum_{i=0}^{N}\frac{S_{i+\frac{3}{2}}-S_{i+\frac{1}{2}}}{h_n}\varphi_i h_n \\
&= -\sum_{i=1}^{N}S_{i+\frac{1}{2}}\frac{\varphi_i-\varphi_{i-1}}{h_n}h_n \\
&= -\int_{-\infty}^{1}S_{\mathcal{T}_n}(x)\frac{\varphi_{\mathcal{T}_n}(x)-\varphi_{\mathcal{T}_n}(x-h_n)}{h_n}dx \\
&= -\left(S_{\mathcal{T}_n},\frac{\varphi_{\mathcal{T}_n}-\varphi_{\mathcal{T}_n}(\cdot-h_n)}{h_n}\right)_{L_2((-\infty,1))}.
\end{aligned} \tag{53}
$$

Function $(S_{\mathcal{T}_n}(\cdot+h_n)-S_{\mathcal{T}_n})/h_n$ is bounded in $L_2(\mathbb{R})$ (see (34)). Then, for any sequence of meshes $(\mathcal{T}_n)_{n\in\mathbb{N}}$ such that $h_n\to 0$ as $n\to\infty$, there exists a subsequence, still denoted by $(\mathcal{T}_n)_{n\in\mathbb{N}}$, such that function $(S_{\mathcal{T}_n}(\cdot+h_n)-S_{\mathcal{T}_n})/h_n$ weakly converges to some function $t(x)$. We also know that $S_{\mathcal{T}_n} \to S$ in $L_2(\mathbb{R})$ as $n \to \infty$.

On the other hand we have that function $\varphi_{\mathcal{T}_n}$ strongly converges to φ and $(\varphi_{\mathcal{T}_n} - \varphi_{\mathcal{T}_n}(\cdot - h_n))/h_n$ strongly converges to $\partial\varphi/\partial x$. Passing to the limit in (53) we obtain:

$$(t(x), \varphi(x))_{L_2((-\infty,1))} = -\left(S(x), \frac{\partial \varphi}{\partial x}\right)_{L_2((-\infty,1))}.$$

By the definition of the weak derivative we obtain that $t(x) = \partial S/\partial x$. Using (34) we have:

$$\left\|\frac{\partial S}{\partial x}\right\|_{L_2((-\infty,1))} \le C_2.$$

We also have that $\partial S/\partial x = 0$ if $x \in (-\infty, 0)$. Hence, the restriction of S to $(0,1)$ is in $H^1((0,1))$. Lemma 4 is proven. $\square$

In order to prove the main convergence theorem we introduce one more assumption.

Assumption 3. *The domain Ω_1, where single-phase flow occurs, consists of a finite number of simply connected subdomains.*

This assumption does not contradict physical meaning of the filtration process.

Theorem 1. *Let Assumptions 1, 2 and 3 be satisfied. For the mesh $\mathcal{T}$ on $(0,1)$ let the pair of vectors $\mathbf{y} = (y_0, y_1, \dots, y_N)^T \in \mathbb{R}^{N+1}$ and $\mathbf{S} = (S_{-\frac{1}{2}}, S_{\frac{1}{2}}, \dots, S_{N+\frac{1}{2}})^T \in \mathbb{R}^{N+2}$ be a solution to (23–32) and let $y_{\mathcal{T}}$ and $S_{\mathcal{T}}$ be defined as*

$$\begin{aligned}
&y_{\mathcal{T}} : (0,1) \to \mathbb{R} \text{ by } y_{\mathcal{T}}(x) = y_i, \text{ if } x \in \mathcal{K}_i, \ \ i = \overline{0, N};\\
&S_{\mathcal{T}} : (0,1) \to (S_*, 1] \text{ by } S_{\mathcal{T}}(x) = S_{i+\frac{1}{2}}, \text{ if } x \in \mathcal{K}_i, \ \ i = \overline{0, N}.
\end{aligned}$$

Then, for any sequence of meshes $(\mathcal{T}_n)_{n\in\mathbb{R}}$ such that $h_n \to 0$, as $n \to \infty$, there exists a subsequence, still denoted by $(\mathcal{T}_n)_{n\in\mathbb{R}}$, such that $y_{\mathcal{T}_n} \to y$ and $S_{\mathcal{T}_n} \to S$ in $L_2((0,1))$, as $n \to \infty$, where $y \in H^1_{0-}((0,1))$ and $S \in H^1((0,1))$ are solutions to the system (20), (21).

Proof. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of meshes on $(0,1)$ such that $h_n \to 0$, as $n \to \infty$. Lemma 3 gives us the existence of solution to the problem (23–32) for any mesh $\mathcal{T}_n$ from sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$. Lemma 4 guarantees that there exists a subsequence, still denoted by $(\mathcal{T}_n)_{n\in\mathbb{N}}$, such that $y_{\mathcal{T}_n} \to y$ and $S_{\mathcal{T}_n} \to S$ in $L_2((0,1))$ as $n \to \infty$ and that $y \in H^1_{0-}((0,1))$ and $S \in H^1((0,1))$.

Following the proof of Theorem 1 from [3] we obtain that $y \in H^1_{0-}$ is a solution to (20) for a any given $S \in L_2((0,1))$. In order to conclude the proof we have to show that $S \in L_2((0,1))$ is a solution to (21) for any given $y \in H^1_{0-}$.

Let $\varphi \in C^\infty([0,1])$ such that $\varphi(0) = 0$. Then the weak formulation (21) can be written in the following way:

$$-T_1 - T_2 + T_3 = 0,$$

where:

$$T_1 = \int_{\Omega_2} \frac{1}{c} \left(y + g(S)\right) \varphi dx, \quad T_2 = \int_0^1 S \frac{\partial \varphi}{\partial x} dx, \quad T_3 = S(1)\varphi(1),$$

where domain Ω_2 is defined in the following way:

$$\Omega_2 = \{x \in (0,1) : S(x) \in (S_*, 1)\}. \tag{54}$$

Taking into account two-step algorithm (27–32) we notice that solution $S_{i-\frac{1}{2}}$ for $i = \overline{0, N+1}$ satisfies:

$$S_{-\frac{1}{2}} = C_0, \tag{55}$$

$$\frac{1}{c}\left(y_i + g(S_{i+\frac{1}{2}})\right) = \frac{1}{h_i^n}(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}), \ \ i \in U, \tag{56}$$

$$S_{i+\frac{1}{2}} - S_* - \eta_n = 0, \ \ i \in F_*, \tag{57}$$

$$S_{i+\frac{1}{2}} - 1 + \eta_n = 0, \ \ i \in F^*; \tag{58}$$

where η_n corresponds to mesh $\mathcal{T}_n$, $U \cup F_* \cup F^* = \overline{0, N}$ and:

$$U = \{i : S_* + \eta_n < \hat{S}_{i+\frac{1}{2}} < 1 - \eta_n\}, \tag{59}$$

$$F_* = \{i : \hat{S}_{i+\frac{1}{2}} \leq S_* + \eta_n\}, \tag{60}$$

$$F^* = \{i : \hat{S}_{i+\frac{1}{2}} \geq 1 - \eta_n\}. \tag{61}$$

We rewrite Eqs. (57,58) in the following way:

$$\frac{1}{h_i^n}\left(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right) + \frac{1}{h_i^n}\left(S_{i-\frac{1}{2}} - S_* - \eta_n\right) = 0, \ \ i \in F_*, \tag{62}$$

$$\frac{1}{h_i^n}\left(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right) + \frac{1}{h_i^n}\left(S_{i-\frac{1}{2}} - 1 + \eta_n\right) = 0, \ \ i \in F^*. \tag{63}$$

Let $\mathcal{T}_n$ be the mesh on $[0,1]$ (see Definition 1), which is one of the meshes of the subsequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ and $\varphi_i = \varphi(x_i)$, $i = \overline{0, N}$. If $\mathbf{S} = \left(S_{-\frac{1}{2}}, S_{\frac{1}{2}}, \ldots, S_{N+\frac{1}{2}}\right)^T$ is a solution to (55–58) for some given $\mathbf{y} = (y_0, y_1, \ldots, y_N)^T$ on the mesh $\mathcal{T}_n$, multiplying (56,62,63) by $\varphi_i h_i^n$ for all $i = \overline{0, N}$ and summing over $i = \overline{0, N}$ we get

$$\begin{aligned} &-\sum_{i\in U} \frac{1}{c}\left[y_i + g(S_{i+\frac{1}{2}})\right] \varphi_i h_i^n + \sum_{i=0}^{N} \frac{1}{h_i^n}\left(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right) \varphi_i h_i^n \\ &+ \sum_{i\in F_*} \frac{1}{h_i^n}\left(S_{i-\frac{1}{2}} - S_* - \eta_n\right) \varphi_i h_i^n + \sum_{i\in F^*} \frac{1}{h_i^n}\left(S_{i-\frac{1}{2}} - 1 + \eta_n\right) \varphi_i h_i^n = 0. \end{aligned} \tag{64}$$

Reordering summation in the second term of (64), we obtain:

$$-\hat{T}_1^n - \hat{T}_2^n + \hat{T}_3^n + \hat{T}_4^n = 0, \tag{65}$$

where:

$$\hat{T}_1^n = \sum_{i\in U} \frac{1}{c}\left[y_i + g(S_{i+\frac{1}{2}})\right]\varphi_i h_i^n, \tag{66}$$

$$\hat{T}_2^n = \sum_{i=0}^{N-1} S_{i+\frac{1}{2}} \frac{\varphi_{i+1}-\varphi_i}{h_i^n} h_i^n, \tag{67}$$

$$\hat{T}_3^n = S_{N+\frac{1}{2}}\varphi_N, \tag{68}$$

$$\hat{T}_4^n = \sum_{i\in F_*} \frac{S_{i-\frac{1}{2}} - S_* - \eta_n}{h_i^n}\varphi_i h_i^n + \sum_{i\in F^*} \frac{S_{i-\frac{1}{2}} - 1 + \eta_n}{h_i^n}\varphi_i h_i^n. \tag{69}$$

Consider the terms in this equation separately. Term $\hat{T}_1^n$ can be written down in the following integral formulation:

$$\hat{T}_1^n = \int_{\Omega_{n,2}} \frac{1}{c}\left(y_{\mathcal{T}_n} + g(S_{\mathcal{T}_n})\right)\varphi_{\mathcal{T}_n} dx,$$

where $\varphi_{\mathcal{T}_n} = \varphi_i$ if $x \in \mathcal{K}_i$, $i = \overline{0,N}$, domain $\Omega_{n,2} = \cup_{i\in U}\mathcal{K}_i$. Using (59), the domain $\Omega_{n,2}$ can be represented in the following form:

$$\begin{aligned}\Omega_{n,2} &= \{\mathcal{K}_i, i : \hat{S}_{i+\frac{1}{2}} \in (S_* + \eta_n, 1-\eta_n)\}\\ &\overset{(32)}{=} \{\mathcal{K}_i, i : S_{i+\frac{1}{2}} \in (S_* + \eta_n, 1-\eta_n)\}\\ &= \{x\in\Omega : S_{\mathcal{T}_n}(x) \in (S_* + \eta_n, 1-\eta_n)\}.\end{aligned} \tag{70}$$

Let us now consider the difference $\left|T_1 - \hat{T}_1^n\right|$:

$$\begin{aligned}\left|T_1 - \hat{T}_1^n\right| &= \left|\int_{\Omega_2} \frac{1}{c}\left(y + g(S)\right)\varphi dx - \int_{\Omega_{n,2}} \frac{1}{c}\left(y_{\mathcal{T}_n} + g(S_{\mathcal{T}_n})\right)\varphi_{\mathcal{T}_n} dx\right|\\ &< \left|\int_{\Omega_2} \frac{1}{c}\left(y + g(S)\right)\varphi dx - \int_{\Omega_{n,2}} \frac{1}{c}\left(y + g(S)\right)\varphi dx\right|\\ &+ \int_{\Omega_{n,2}} \left|\frac{1}{c}\left(\left(y_{\mathcal{T}_n} + g(S_{\mathcal{T}_n})\right)\varphi_{\mathcal{T}_n} - \left(y + g(S)\right)\varphi\right)\right| dx\\ &\le \left|\int_{\Omega_2\oplus\Omega_{n,2}} \frac{1}{c}\left(y + g(S)\right)\varphi dx\right|\\ &+ \int_{\Omega} \left|\frac{1}{c}\left(\left(y_{\mathcal{T}_n} + g(S_{\mathcal{T}_n})\right)\varphi_{\mathcal{T}_n} - \left(y + g(S)\right)\varphi\right)\right| dx.\end{aligned} \tag{71}$$

The second term on the right-hand side in inequality (71) converges to zero as $n \to \infty$. To prove that the first term on the right-hand side also converges to zero as $n \to \infty$ we have to show that for any $\varepsilon > 0$ there exists $N^\varepsilon \in \mathbb{N}$ such that:

$$\int_{\Omega_2\oplus\Omega_{n,2}} dx < \varepsilon \text{ for all } n \ge N^\varepsilon. \tag{72}$$

This condition is sufficient, since all functions under the integral in (71) are bounded.

It was shown in Lemma 4 that $S_{\mathcal{T}_n} \to S$ in $L_2((0,1))$, which means that it also converges in measure. Then, for any $\delta > 0$ and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that:

$$\operatorname{meas}\left(\{x \in \Omega : |S_{\mathcal{T}_n}(x) - S(x)| > \delta\}\right) < \epsilon, \text{ for all } n \geq N,$$

where $\Omega = (0,1)$. To simplify the notations we suppose that $\delta = \epsilon$ then definition of convergence in measure yields for any $\delta > 0$ there exists $N = N(\delta) \in \mathbb{N}$ such that:

$$\operatorname{meas}\left(\{x \in \Omega : |S_{\mathcal{T}_n}(x) - S(x)| > \delta\}\right) < \delta, \text{ for all } n \geq N. \tag{73}$$

Using (73) we define two subsets of Ω in the following way:

$$\Omega^\delta = \{x \in \Omega : |S_{\mathcal{T}_n}(x) - S(x)| \leq \delta\}, \tag{74}$$

$$\tilde{\Omega}^\delta = \Omega \setminus \Omega^\delta = \{x \in \Omega : |S_{\mathcal{T}_n}(x) - S(x)| > \delta\}. \tag{75}$$

Next we consider the integral (72). Using an indicator function, namely $\mathbf{1}_A(x) = 1$, if $x \in A$, and $\mathbf{1}_A(x) = 0$, if $x \notin A$, we obtain:

$$\int_{\Omega_2 \oplus \Omega_{n,2}} dx = \int_\Omega \mathbf{1}_{\Omega_2 \oplus \Omega_{n,2}} dx = \int_\Omega \mathbf{1}_{\Omega_2 \setminus \Omega_{n,2}} dx + \int_\Omega \mathbf{1}_{\Omega_{n,2} \setminus \Omega_2} dx. \tag{76}$$

Using sets Ω^δ and $\tilde{\Omega}^\delta$, defined in (74,75), we split integrals on the right hand side of (76) into four integrals:

$$\int_{\Omega_2 \oplus \Omega_{n,2}} dx = I_1 + I_2 + I_3 + I_4, \tag{77}$$

$$I_1 = \int_\Omega \mathbf{1}_{(\Omega_2 \cap \Omega^\delta) \setminus (\Omega_{n,2} \cap \Omega^\delta)} dx, \tag{78}$$

$$I_2 = \int_\Omega \mathbf{1}_{(\Omega_2 \setminus \Omega_{n,2}) \cap \tilde{\Omega}^\delta} dx, \tag{79}$$

$$I_3 = \int_\Omega \mathbf{1}_{(\Omega_{n,2} \cap \Omega^\delta) \setminus (\Omega_2 \cap \Omega^\delta)} dx, \tag{80}$$

$$I_4 = \int_\Omega \mathbf{1}_{(\Omega_{n,2} \setminus \Omega_2) \cap \tilde{\Omega}^\delta} dx. \tag{81}$$

Summing integrals I_2 and I_4 we have:

$$I_2 + I_4 = \int_\Omega \mathbf{1}_{(\Omega_2 \oplus \Omega_{n,2}) \cap \tilde{\Omega}^\delta} dx < \int_\Omega \mathbf{1}_{\tilde{\Omega}^\delta} < \delta.$$

Let us now consider the integral I_1. Introducing a set:

$$\tilde{\Omega}_2 = \{x \in \Omega : S(x) \in (S_* + \delta + \eta_n, 1 - \delta - \eta_n)\},$$

It is easy to show that $\left(\tilde{\Omega}_2 \cap \Omega^\delta\right) \subseteq \left(\Omega_{n,2} \cap \Omega^\delta\right)$. Then, we notice that:

$$I_1 \leq \int_\Omega \mathbf{1}_{(\Omega_2 \cap \Omega^\delta) \setminus \left(\tilde{\Omega}_2 \cap \Omega^\delta\right)} dx = \int_\Omega \mathbf{1}_{\left(\Omega_2 \setminus \tilde{\Omega}_2\right) \cap \Omega^\delta} dx,$$

This integral is the measure of the domain where $|S_{\mathcal{T}_n}(x) - S(x)| \leq \delta$ and:

$$S \in (S_*, S_* + \delta + \eta_n] \cup [1 - \delta - \eta_n, 1). \tag{82}$$

Using Assumption 2 for small enough δ and η_n we notice that $I_1 \to 0$ as $n \to \infty$.

Concerning I_3, we choose δ such that $\delta < \eta_n$. Then, we have:

$$\left(\Omega_{n,2} \cap \Omega^{\delta}\right) \subseteq \left(\Omega_2 \cap \Omega^{\delta}\right)$$

and it follows that $I_3 = 0$.

Now we consider the term $\hat{T}_2^n$. Thanks to the regularity of the function φ, we notice, that:

$$\frac{\varphi_{i+1} - \varphi_i}{h_n} = \left.\frac{\partial \varphi}{\partial x}\right|_{x_i} + R_i, \text{ where } |R_i| \leq C_1 h_n, \ \ i = \overline{0, N-1},$$

where C_1 is a some constant, which depends only on φ. Therefore, Eq. (67) is split into two parts:

$$\hat{T}_2^n = \hat{T}_{2,1}^n + \hat{T}_{2,2}^n, \tag{83}$$

$$\hat{T}_{2,1}^n = \sum_{i=0}^{N-1} S_{i+\frac{1}{2}} \left.\frac{\partial \varphi}{\partial x}\right|_{x_i} h_n, \tag{84}$$

$$\hat{T}_{2,2}^n = \sum_{i=0}^{N-1} S_{i+\frac{1}{2}} R_i h_n. \tag{85}$$

Eq. (84) can also be written in the following way:

$$\hat{T}_{2,1}^n = \int_0^{1-h_n/2} S_{\mathcal{T}_n} \left(\frac{\partial \varphi}{\partial x}\right)_{\mathcal{T}_n} dx,$$

where $\left(\frac{\partial \varphi}{\partial x}\right)_{\mathcal{T}_n} = \left.\frac{\partial \varphi}{\partial x}\right|_{x_i}$, if $x \in \mathcal{K}_i$, $i = \overline{0, N-1}$. Finally, we have

$$\hat{T}_{2,1}^n \to T_2 \text{ as } n \to \infty.$$

Since all values $S_{i-\frac{1}{2}}$ are bounded, then (85) yields:

$$\hat{T}_{2,2}^n \to 0 \text{ as } n \to \infty.$$

Using (68) we remark, that:

$$\hat{T}_3^n \to T_3 \text{ as } n \to \infty,$$

Now let us consider $\hat{T}_4^n$ defined by (69). We notice that sets F_* and F^* can be divided into the following subsets:

- $F_{*,U\to F_*} = \{i \in F_* : S_* + \eta_n < \hat{S}_{i-\frac{1}{2}} < 1 - \eta_n\}$,
- $F_{*,F^*\to F_*} = \{i \in F_* : \hat{S}_{i-\frac{1}{2}} \geq 1 - \eta_n\}$,
- $F_{*,internal} = \{i \in F_* : \hat{S}_{i-\frac{1}{2}} \leq S_* + \eta_n\}$,
- $F^*_{U\to F^*} = \{i \in F^* : S_* + \eta_n < \hat{S}_{i-\frac{1}{2}} < 1 - \eta_n\}$,
- $F^*_{F_*\to F^*} = \{i \in F^* : \hat{S}_{i-\frac{1}{2}} \leq S_* + \eta_n\}$,

- $F^*_{internal} = \{i \in F^* : \hat{S}_{i-\frac{1}{2}} \geq 1 - \eta_n\}$.

Then, term $\hat{T}^n_4$ yields:

$$\hat{T}^n_4 = \sum_{i \in F_1} \frac{S_{i-\frac{1}{2}} - S_* - \eta_n}{h^n_i} \varphi_i h^n_i + \sum_{i \in F_2} \frac{S_{i-\frac{1}{2}} - 1 + \eta_n}{h^n_i} \varphi_i h^n_i,$$

$$F_1 = F_{*,U \to F_*} \cup F_{*,F^* \to F_*}, \quad F_2 = F^*_{U \to F^*} \cup F^*_{F_* \to F^*},$$

since for all $i \in F_{*,internal}$ saturation $S_{i-\frac{1}{2}}$ is equal to constant S_* and for all $i \in F^*_{internal}$ saturation $S_{i-\frac{1}{2}}$ is equal to constant 1.

Using inequalities (36–38) for $\hat{T}^n_4$ we have:

$$\left|\hat{T}^n_4\right| \leq \sum_{i \in F_1 \cup F_2} \frac{\left|S_{i-\frac{1}{2}} - \hat{S}_{i+\frac{1}{2}}\right|}{h^n_i} |\varphi_i| h^n_i \leq \frac{C_1 + g^*}{c} \varphi^* h_n \sum_{i \in F_1 \cup F_2} 1,$$

where function φ is bounded by constant $\varphi^* > 0$ since it is a continuous function on $[0, 1]$.

Since domain Ω_1 consists of a finite number of simply connected subdomains and each subdomain corresponds to one element of set $F_1 \cup F_2$, the number of elements of set $F_1 \cup F_2$ does not depend on discretization. Then, we obtain:

$$\left|\hat{T}^n_4\right| \to 0 \text{ as } n \to \infty. \tag{86}$$

Hence, the theorem is proven. □

5 Numerical experiment

The goal of this section is to investigate numerically some typical examples. We are going to estimate the rate of convergence of the proposed numerical scheme (23–32) and verify the assumptions made in the previous sections. The test problems are selected such that fully saturated regions appear.

For the numerical experiments we consider three different cases of parameters which are typical for a paper layer during a production process. All information on input data is presented in Table 1 and in Fig. 2 and 3. Note that presented data satisfy Assumption 1. Obtained distributions of saturation and ´ pressure are presented in Fig. 4 and 5, respectively.

Exact solutions to the presented problems are unknown. To obtain the convergence rate, the reference solutions, by which the errors are measured, has been calculated on a very fine mesh $\mathcal{T}_*$. Corresponding distributions of saturation and pressure are denoted by $S_{\mathcal{T}_*}$ and $p_{\mathcal{T}_*}$. Then we define the relative error E_n between the discrete solution $S_{\mathcal{T}_n}$, $p_{\mathcal{T}_n}$ and the reference solution $S_{\mathcal{T}_*}$, $p_{\mathcal{T}_*}$ as:

$$E_n = \left(\frac{\|S_{\mathcal{T}_*} - S_{\mathcal{T}_n}\|^2_{L_2(\Omega)}}{\|S_{\mathcal{T}_*}\|^2_{L_2(\Omega)}} + \frac{\|p_{\mathcal{T}_*} - p_{\mathcal{T}_n}\|^2_{L_2(\Omega)}}{\|p_{\mathcal{T}_*}\|^2_{L_2(\Omega)}} \right)^{\frac{1}{2}}.$$

For each test case we consider three different values of $\eta_{\mathcal{T},i}$:

$$\eta_{\mathcal{T},i} = \frac{C_i h}{\operatorname{meas} \Omega}, \quad i = 1, 2, 3,$$

Table 1: Experimental Data

Variable	Test Case 1	Test Case 2	Test Case 3	Dimension
C_0	50	60	55	$[\%]$
c	16.7	200	125	$[Pa \cdot m]$
k		$S^{3.5}$		$-$
S_r		0.1		$-$
S_*		$S_r + \varepsilon$		$-$
p_c^{stat}		$a(\phi - 1)\left(\frac{1}{S-S_r} - \frac{1}{1-S_r}\right)^{1/2}$		$[Pa]$
a		$\frac{P_0}{1-\phi_0}\left(\frac{1}{C_0-S_r} - \frac{1}{1-S_r}\right)^{-1/2}$		$[Pa]$
P_0		-5000		$[Pa]$
ϕ_0		87.5		$[\%]$
Ω		$(-0.05, 0.05)$		$[m]$

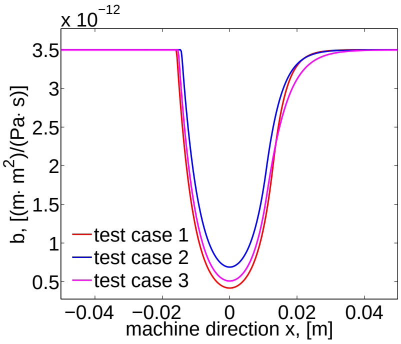

Fig. 2: Input function $b(x)$

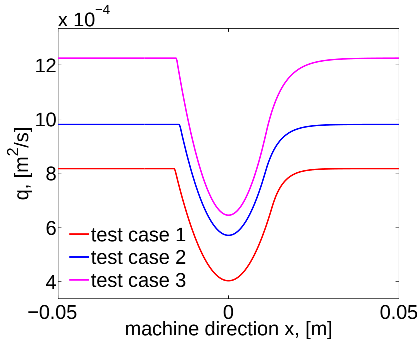

Fig. 3: Input function $q(x)$

where $C_1 = 1$, $C_2 = 2$ and $C_3 = 10$. The results are given in Fig. 6. For all three cases and different values of $\eta_{\mathcal{T},i}$ we observe a first-order convergence (the estimated order r is defined as:

$$r = \frac{1}{N_e - 2} \sum_{n=2}^{N_e - 1} \frac{\log |E_{n+1}/E_n|}{\log |E_n/E_{n-1}|},$$

where N_e is the number of experiments).

In the proof of Theorem 1 we have obtained that the parameter η can not be too small, because the convergence of the measure of the domain with single-phase flow regime depends on it. So we carry out numerical experiments to estimate the behavior of the domain measure convergence for different values of η. The reference domain with single-phase water flow is denoted by $\Omega_{*,1}$. Then, error M_n between the measure of the reference domain $\Omega_{*,1}$ and the measure of the domain for a current mesh $\Omega_{n,1}$ is computed as:

$$M_n = \frac{|\operatorname{meas}(\Omega_{*,1}) - \operatorname{meas}(\Omega_{n,1})|}{\operatorname{meas}(\Omega_{*,1})}.$$

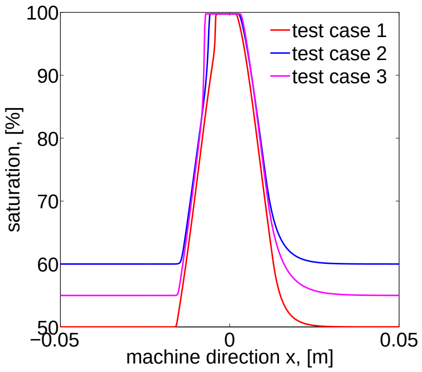

Fig. 4: Distribution of saturation

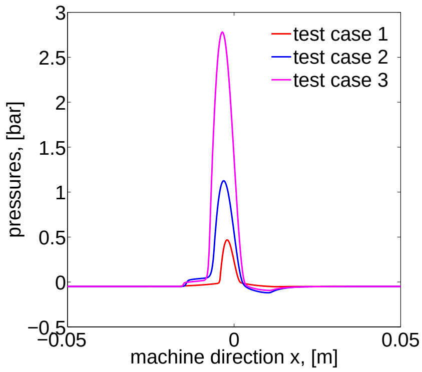

Fig. 5: Distribution of pressure

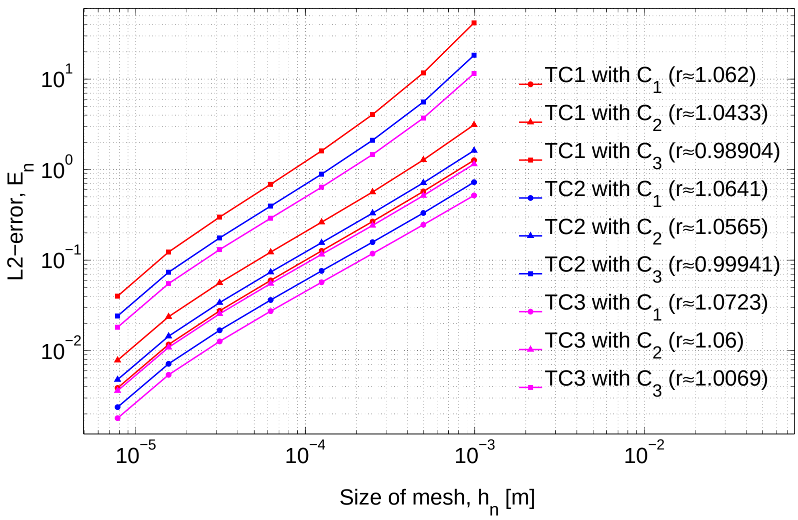

Fig. 6: Convergence of the solution (TC is an abbreviation for "test case")

Results are presented in Fig. 7. As it follows from the proof of Theorem 1 the optimal value of parameter η is unknown in advance. The results of numerical experiments show that the convergence of the solution is not sensitive to the value of η (see Fig. 6). But here we should take into account that increasing η we also increase the solution error. On the other hand, convergence of measure of the single-phase flow domain shows stronger behavior for bigger values of η (see Fig. 7).

The last goal of numerical experiments was to verify Assumption 2. Since the exact solution is unknown, we use the reference solution $S_{\mathcal{T}_*}$ and plot in Fig. 8 the dependence of δ_ϵ on ϵ from condition (22). It follows from the presented results that Assumption 2 is satisfied for given numerical examples.

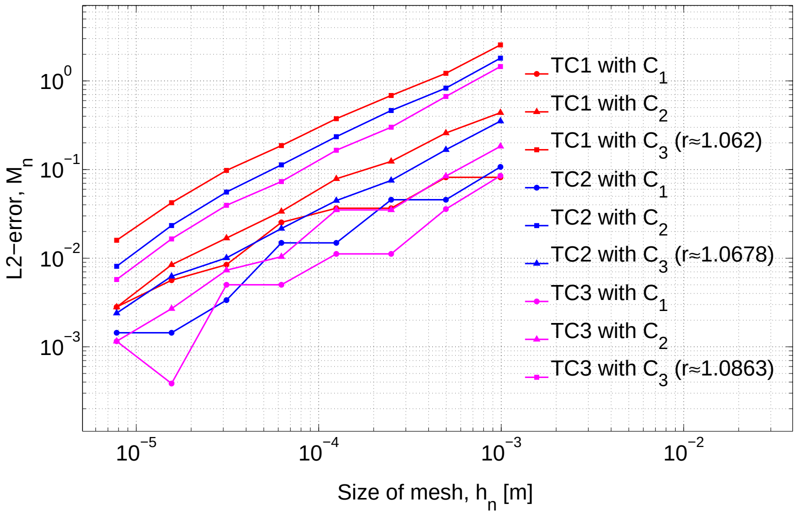

Fig. 7: Convergence of the domain measure (TC is an abbreviation for "test case")

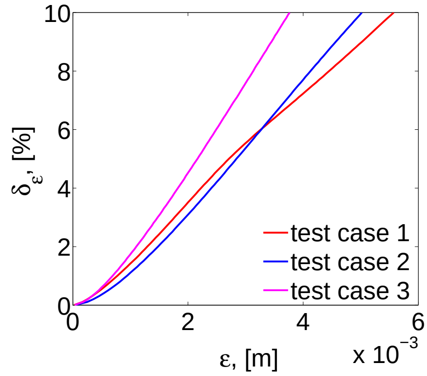

Fig. 8: Verification of Assumption 2

6 Conclusion

The objective of this work is to show the convergence of the numerical solution to the continuous one for the system of equations describing the pressing section of a paper machine including the dynamic capillary effect. One of the challenges of this problem is an evaluation of the fully saturated regions. Solving this problem we have to keep in mind that in the computational domain the region with single-phase water flow may appear. At first we state two mathematical models for the both flow regimes with a free boundary. Then, we combine them into one model in the whole computational domain. For the discretized system we propose a numerical algorithm, which implicitly takes into account the two flow regimes.

The theoretical part of this work contains the proof of existence of solution to the discrete system, compactness and the convergence theorem. The main idea of the theoretical studies is to prove the convergence for the input data which is typical for real numerical experiments. Since we can not imply too strong assumptions we do not get precise estimates on the convergence order and we are not concerned with the proof of uniqueness.

Some assumptions for solution, which are made during the theoretical studies, are verified by the numerical experiment. We also have estimated numerically the rate of convergence of solution and measure of the fully saturated region.

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